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(Preliminary)

ON
NOISE DISTORTION
AND
HARMONIC DISTORTION
MEASUREMENTS

PROJECT 10350

FOR
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ABSTRACT

A desirable distortion measurement would correlate with speech intelligibility in addition to indicating the magnitude of harmonics and other frequencies not in the initial signal. Output frequencies not present at the input are a measure of non-linearity. Among the more commonly used test signals are single sine waves (harmonic distortion) and two sine waves (intermodulation distortion). This report gives a comparison between these distortion measures and proposed distortion measurements using wide band noise test signals. Mathematical analysis shows that selection of test signal form is relatively unimportant to determination of the non-linear properties of an element. Experimental evaluation of distortion measurement with sine waves and noise was carried out on four non-linear elements: full wave rectification, half wave rectification, infinite peak clipping, and antisymmetric square law. The distortion elements considered here represent a subset of the total set of possible distortion conditions which a distortion measuring instrument should correlate with intelligibility. Sine wave and noise signals ordered these devices the same. This order was different from the order obtained by Licklider, et. al., for monosyllable speech intelligibility. Spectrum shaping by differentiation or integration either before or after the distortion process alters both of the distortion measures and also intelligibility; however, the ordering is still different and in some combinations intelligibility and distortion changes are contradictory.

the same property of a non-linear device: its ability to produce new frequencies. Production or creation of new frequencies by distortion means that some or all of the energy at particular frequencies of the input signal is redistributed to other parts of the spectrum. The magnitude of new frequency energy is easily measured with distortion meters or wave analyzers when the signal consists of one or two sine waves. When the test signal is a noise, the "new frequencies" can be detected only by first creating a window or band void of signal energy in the noise, the new frequencies still being observed with a wave analyzer or selective filter. The instrumentation to carry-out N.D. is thus complicated by addition of a narrow band rejection filter of high quality. If N.D. is to be observed at a number of frequencies, the rejection filter must be variable and track with a narrow band analyzer used to observe the distortion components, or a number of fixed filters must be used. Unfortunately, N.D. results depend on how the test signal is generated so that standardization of N.D. must include detailed instrumentation specifications.

Frequency response shaping affects each of the distortion measures differently. Therefore, direct correlation between the methods of measurement is not always possible.

Licklider, et. al.,[4,5,6] reported speech intelligibility scores for the interaction of frequency response shaping with non-linear distortion. Using these scores, we find that neither H.D. nor N.D. rank order different non-linear distortions according to word intelligibility. Also, the combined effect of non-linear distortion and frequency response shaping on word intelligibility is not predicted by H.D. or N.D. measurements.

II. Comparison of H.D., I.M., N.D.

Speech communication systems and components are usually rather complex, having irregular frequency responses and one or more non-linear transfer elements. Considerable insight into the behavior of these systems can be obtained by examining the model in Fig. 1. The model consists of a linear but frequency dependent input circuit having transfer function, A(f); followed by a memory free frequency independent non-linear element; followed by a linear but frequency dependent output circuit with transfer function, B(f). These sub-components are independent of each other.

We first consider only the effect of the non-linearity. This corresponds to making A(f) and B(f) constants, i.e., flat input and output. Table I shows H.D., I.M., and N.D. relationships for a second order power series ($av + bv^2$) non-linear transfer function where a and b are constants independent of frequency and time. Similarly, relationships can be obtained for higher order non-linearities.

The SMPTE and CCIF methods measure I.M.; H.D. is here the ratio of the harmonic output components to the total output, sometimes called total harmonic distortion.

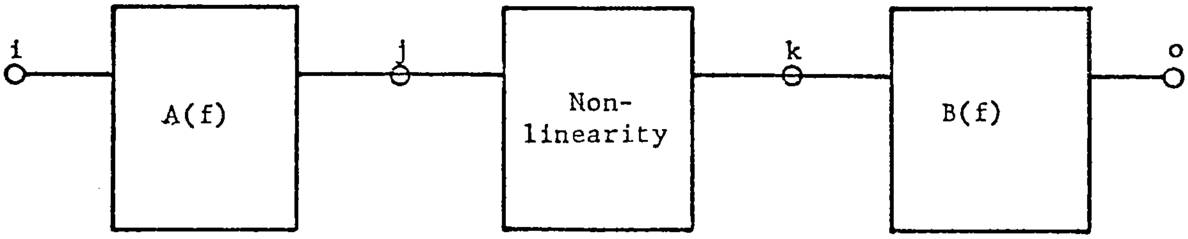


FIG. (1)

TABLE I

Method	% Distortion
H.D.*	$\frac{b}{a}\frac{A_1}{\sqrt{2}}\frac{1}{\sqrt{1+\frac{b^2A_1^2}{2a^2}}}$
SMPTE*	$\frac{b}{a}2\sqrt{2}A_2$
CCIF*	$\frac{b}{a}\frac{A_3}{\sqrt{2}}$
N.D.**	$\frac{b}{a}\sqrt{2}A_4\frac{1}{\sqrt{1+\frac{2b^2A_4^2}{a^2}}}$

A_1 = Rms of drive

A_2 = Rms of lower frequency drive

A_3 = Rms of either drive frequency

A_4 = Rms of white noise

*Refs. 7 and 8.

**Appendix I this report. The result is given for passively filtered noise.

The relationships in Table I emphasize that there is no fundamental difference among the methods. Each method measures the ability of non-linear devices to create new frequencies. Each of the measurements is sufficient to specify the non-linear function. In this example, the ratio of coefficients a and b is uniquely determined by the four measures. Although numerical values will be different for N.D. and H.D., their functional dependence on the coefficients a and b is similar.

On the basis of this analysis only, we may conclude that since N.D. and H.D. and I.M. measure the same properties of a non-linearity, no preference can be given for any one as a speech intelligibility predictor. This is suggested by the fact that H.D. and I.M. have not been satisfactory predictors.

When the input and output circuits introduce arbitrary frequency response shaping, the fundamental equivalence of these methods can be obscured. Appendix I shows that for non-linear systems representable by power series, N.D. is insensitive to filtering or spectrum shaping following the distorting element. The reason for this insensitivity can be appreciated from the discussion of the N.D. measurement procedure in Sections III and IV, following. It arises because the distortion energy and the reference total signal energy are measured at the same frequency. It can be shown that H.D. and I.M. are sensitive to spectrum shaping following a non-linearity. In these two distortion measurement methods, the distortion energy appears and is measured at frequencies different from the input signal frequency. Also, it can be shown that H.D. is insensitive, that I.M. may or may not be sensitive and that N.D. is sensitive to input spectrum shaping. The various effects of spectrum shaping notwithstanding, all three techniques give measures of the new frequencies and redistribution of energy created in the non-linearity. All three measurements are equivalent in this respect.

Unlike frequency independent systems, a change in test frequency or spectrum can alter the test result for a given system. This causes a serious problem in interpreting results for systems having highly irregular frequency responses.

III. N.D. Procedure

The test signal for N.D. measurement is a wideband noise with a narrow stop band. Stop bandwidth relative to pass bandwith must be specified, since, increasing the stop bandwidth removes frequency components that would otherwise be available for generating distortion products. Pederson[2] showed that the ratio of stop band to the passband width of the system under test should not exceed 10% for 10% accuracy. Band stop attenuation must be high if small values of distortion are to be measured. A "hole" depth of 40 dB is equivalent to 1% distortion. Hole depths greater than 40 dB are necessary to resolve a distortion of 1%. Finally, the full utilization of the hole depth requires a detector with low internal noise.

Two methods for implementing the N.D. test have been evaluated: passive filtering[1], and heterodyning of the noise in a single sideband suppressed carrier circuit[3]. The methods produce different test signal characteristics.

Passively Filtered White Noise

Following Bang's technique[1], a white noise signal was filtered with a ten element LC filter to produce a uniform spectrum except for a 50 dB deep stop band centered at 2 K Hz. The 6 dB down points were 300 Hz apart (Fig.2). A General Radio 1900-A wave analyzer having a 50 Hz bandpass was used to measure components at 2 K Hz.

Distortion measurements were made in two steps. First, the white noise source was applied directly to the system under test, i.e., no stop band. If the system was linear, the wave analyzer measured the linearly transferred components within the 50 Hz band at 2 K Hz. However, if distortion was present, some of the distortion products resulting from harmonic generation and intermodulation of components also fell in this narrow band. Thus the measured output was the sum of linear and distortion products. Second,

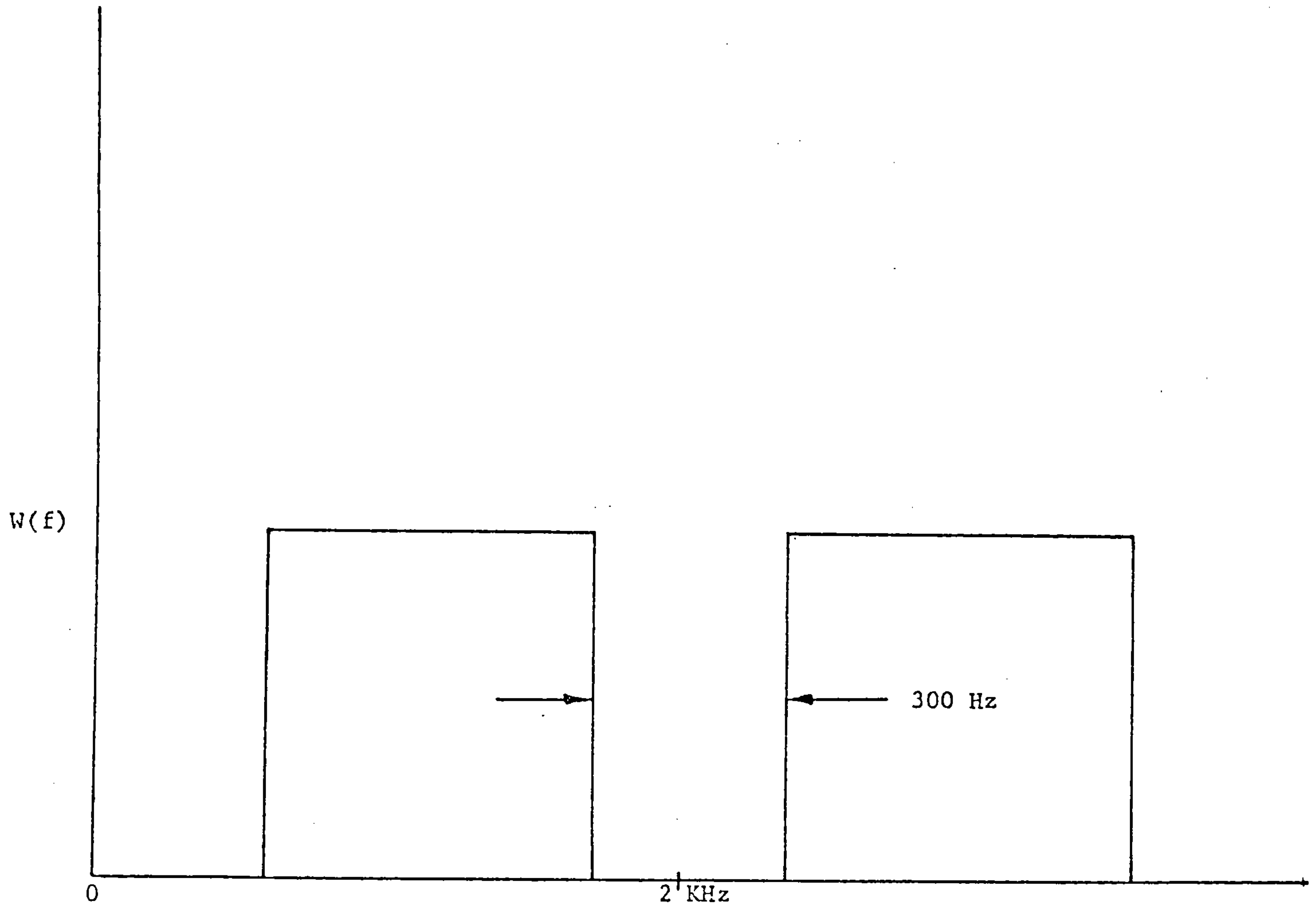


FIG. (2)

the notched spectrum, i.e., the noise with a narrow stopband, was applied at the same drive level (i.e., the same Rms voltage in a 200-4000 cps band). If the system was linear, "no signal" was measurable since no signal was impressed in the band. However, if distortion was present, some of the distortion products appeared in the notch band and were measured. "Noise distortion" was expressed as a percentage based on the ratio of the signal at the notch frequency with and without the notch in the test signal.

An important feature of the passive filter technique is that the noise spectrum retains the statistics of the white noise source. The only effect of the filter is removal of a narrow band of frequencies.

Heterodyne Method

Since it may be desirable to determine the noise distortion at frequencies other than 2.0 K Hz, either a number of stopband filters or a tunable stopband filter scheme is required. Burnett[3] suggested a heterodyne technique that generates a notched spectrum. The notch frequency can be swept over a wide range and synchronously locked to a detector and recorder. However, noise signal processing reduces test signal randomness.

Distortion measurement by N.D. is sensitive to signal randomness. Therefore a detailed discussion of the heterodyne scheme follows.

A wide band white noise signal was sharply filtered to produce a "box car" spectrum from 150 Hz to 5 K Hz (Fig. 3). This signal was amplitude modulated with a high frequency signal $f_m + f_h$ to produce first order sidebands. (Fig.4). A band pass filter removed higher order sidebands and sharply filtered out the portion of the lower sideband below frequency f_m (Fig. 5). Next, the signal was amplitude modulated at frequency f_m to shift the signal down as shown in Fig. 6. This constituted the test signal having a 300 Hz wide stopband or notch located at f_h. Components shown in Fig. 6, symmetrical about the notch frequency, although different in frequency, come from the same instantaneous

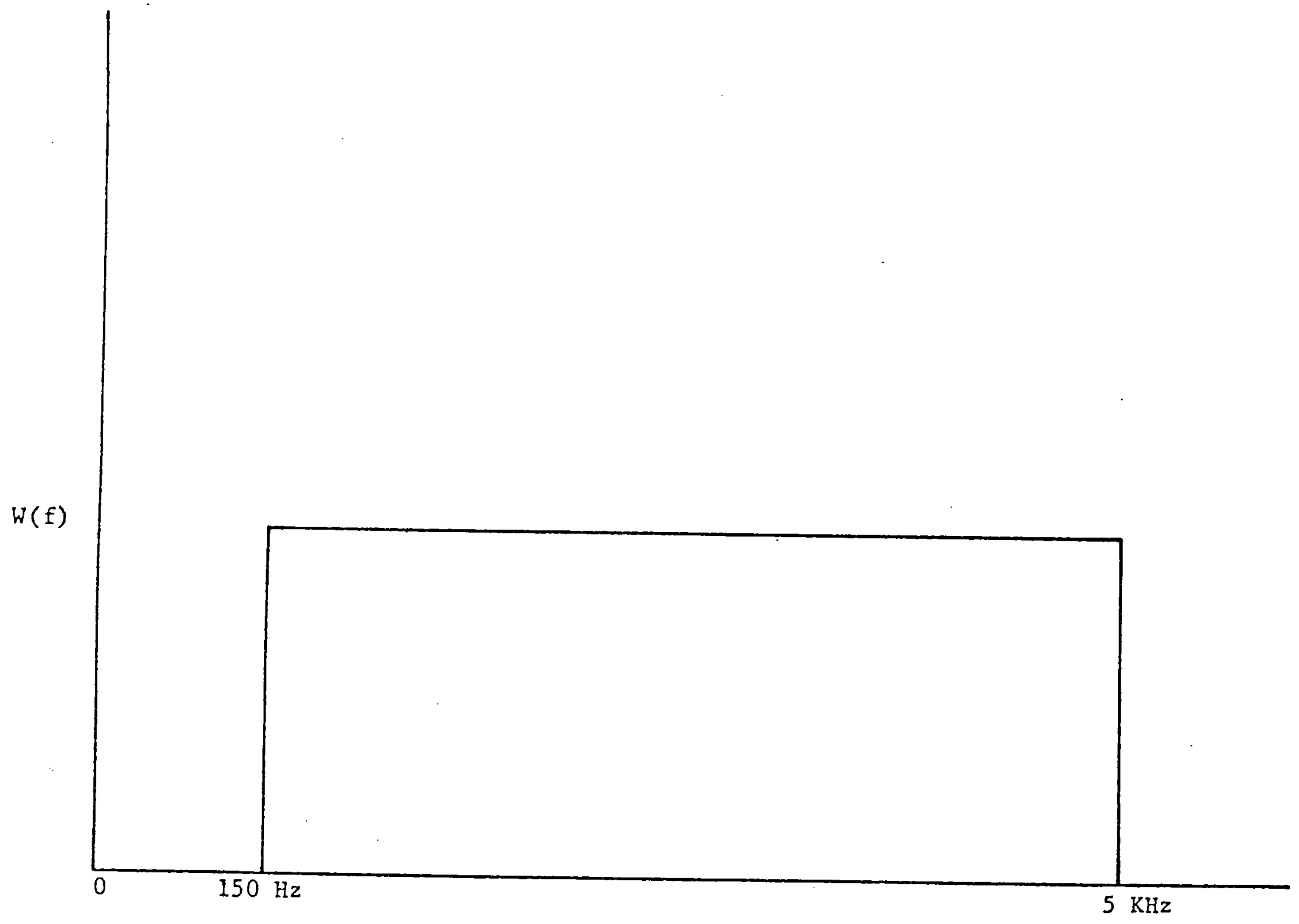


FIG. (3)

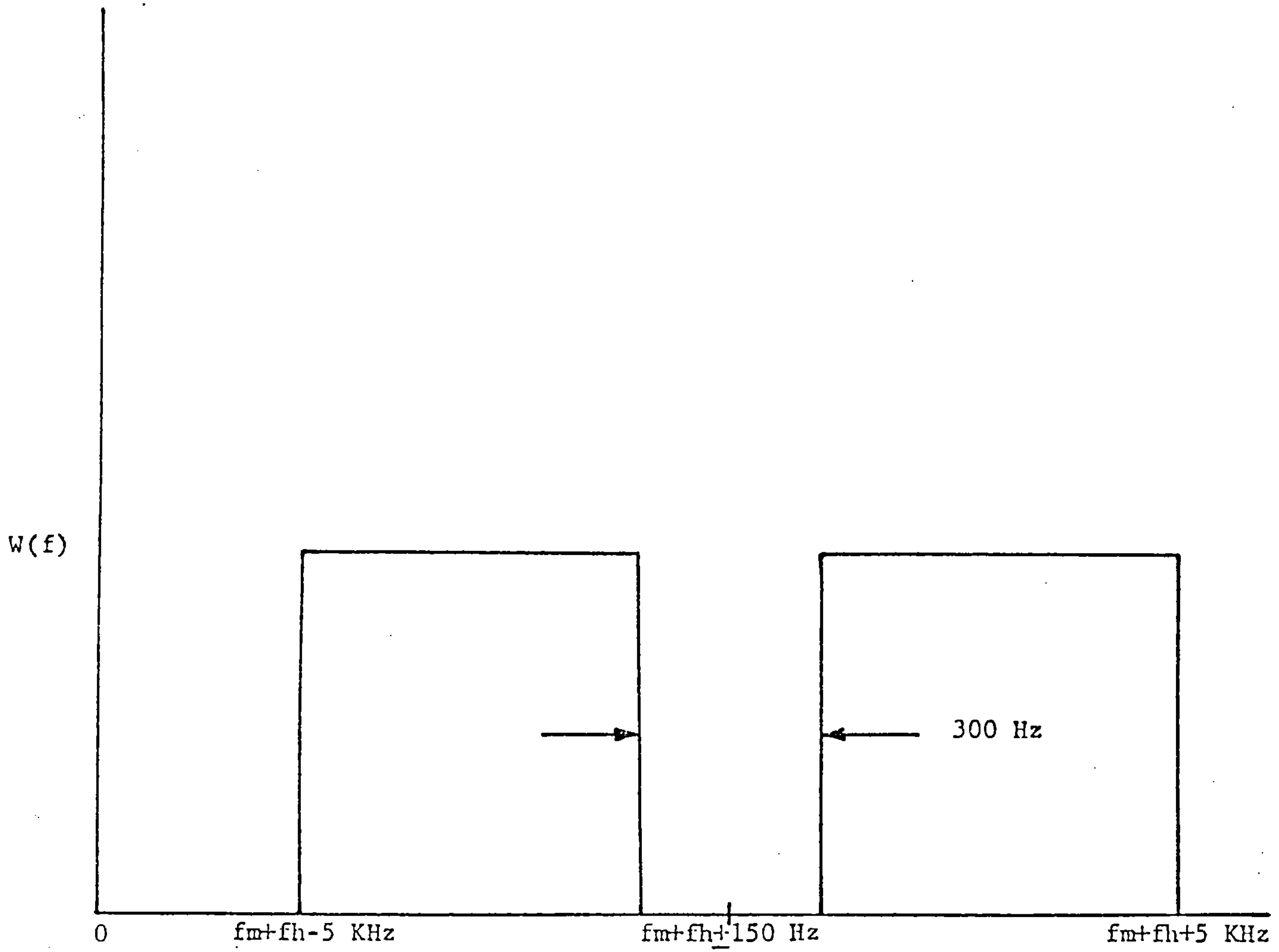


FIG. (4)

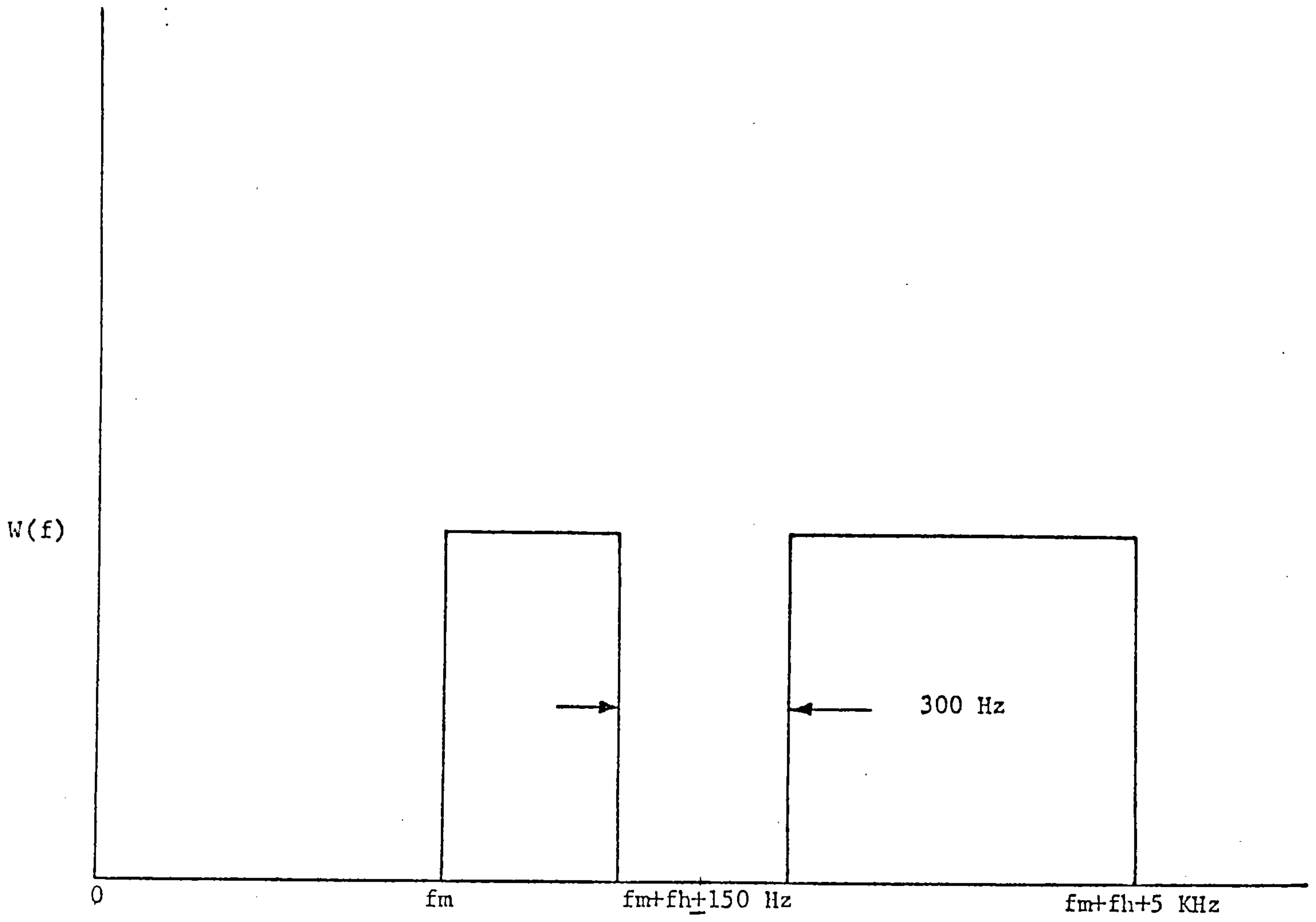


FIG. (5)

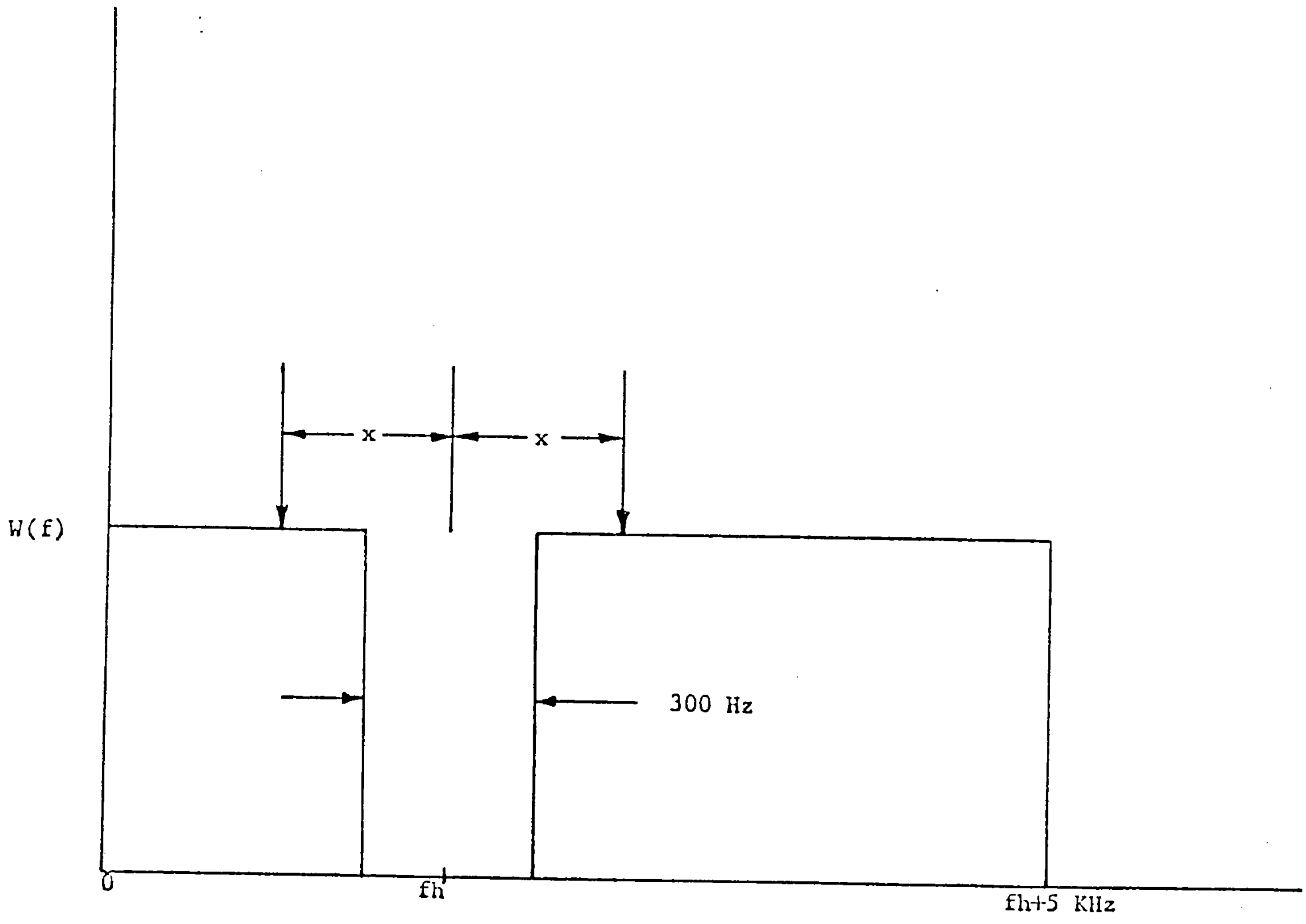


FIG. (6)

frequency component in the original white noise. Correlation between components above and below the notch frequency is produced during the first modulation step.

The fraction of components related depends on the relative contribution of the upper and lower sideband to the total signal in the system passband. The maximum number of correlated components occurs when the hole is in the center of the passband, i.e., equal contributions from upper and lower sidebands. The number of correlated components approaches zero as the hole moves out of the passband, i.e., only one sideband present. Differences between N.D., of from 8% to 12% (Table IV), observed with passive filter and heterodyne generated notch signals, may be due to this signal correlation.

A mathematical analysis of the signals generated by passively filtering a noise and by heterodyning shows that the former is a stationary ensemble. The latter is a non-stationary ensemble as the result of the modulation processes.

An additional consequence of the heterodyne method is the generation of a single frequency component, f_h, in the center of the notch. This component results from imperfect suppression of the modulating frequency $f_m + f_h$ in the first modulation step. Although the detector avoids measuring it, this component is present in the test signal and available for generating distortion products. This component's effective spectral density should not exceed the average spectral density of the noise. (Balanced modulators were necessary in our copy of Burnett's equipment to reduce this component magnitude to the average spectral density of the noise.)

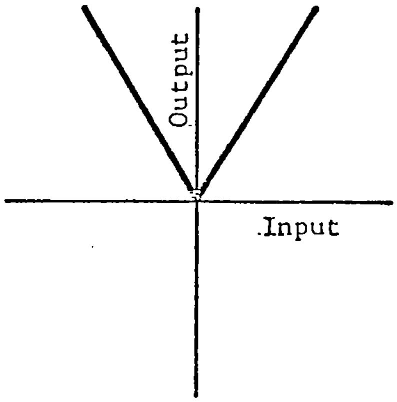
IV. Distortion Test Devices

Four non-linear devices were selected for this study. Each device was amplitude independent, frequency independent and had zero memory. The amplitude independent feature fixes the theoretical H.D. for each device; FWR (full wave rectification) 100%, HWR (half wave rectification) 40%, IPC (nearly infinite peak clipping) 36%, ASL (antisymmetrical square law) 20%. Transfer functions of these non-linearities are shown in Fig. 7. Licklider, et. al., studied word intelligibility for FWR, HWR and IPC. ASL represents a form of cross-over distortion, i.e., mild center clipping which is independent of signal amplitude. Although the selected devices are frequency independent, experimental tests included frequency shaping of the test signal before and after distortion. Memory effects were not studied.

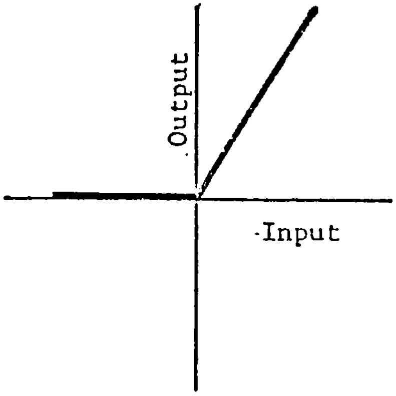
These devices were chosen for the following reasons: (1) ease of construction; (2) unambiguous specification of distortion device; (3) unambiguous signal level settings; (4) availability of documented subjective test results; (5) representative of classes of transfer characteristics; (6) analytical tractability for calculation of H.D., I.M. and N.D.; and (7) a basis for understanding more complicated distortion mechanisms.

V. Test Results

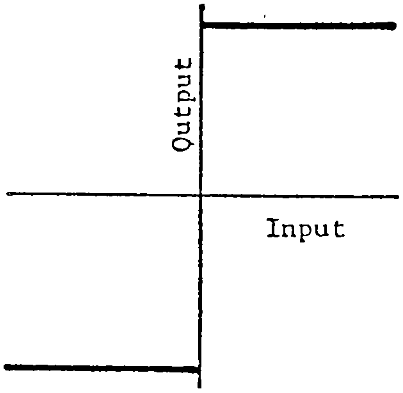
Fig. 8, shows the signal flow diagram of the experimental set-up. Four signal sources were used: 500 cps sine wave, recorded speech, heterodyned white noise spectrum and passively filtered white noise spectrum. The input was pass band limited to 300-4000 Hz. The signal could be directly applied to the non-linear devices or frequency shaped by prior differentiation or integration. Differentiation and integration are mathematical operations that describe a common type of spectrum shaping. When applied to a broad band spectrum differentiation changes the spectrum shape so that amplitude varies by +6 dB per octave. Integration changes the



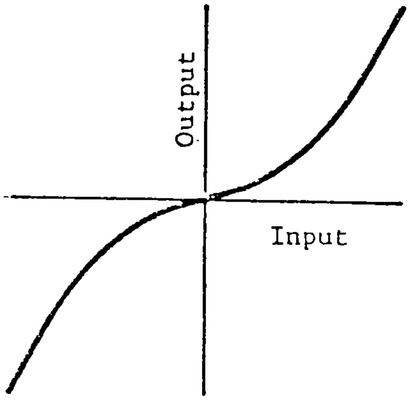
(a) Full Wave Rectification FWR



(b) Half Wave Rectification HWR



(c) Infinite Peak Clipper IPC



(d). Antisymmetrical Square Law ASL

FIG. (7)

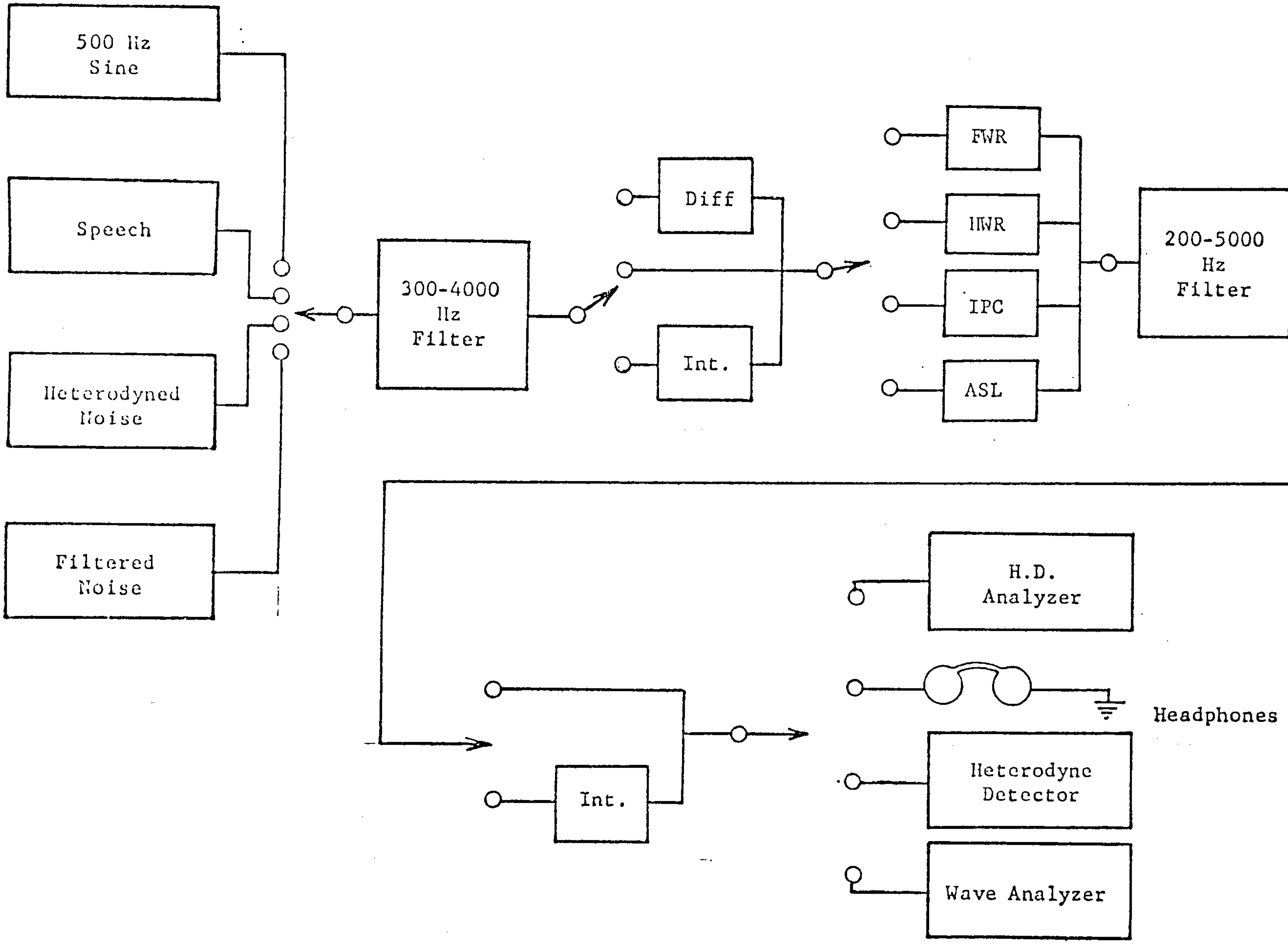

500 Hz Sine
Speech
Heterodyned Noise
Filtered Noise
300-4000 Hz Filter
Diff
Int.
FWR
HWR
IPC
ASL
200-5000 Hz Filter
H.D. Analyzer
Headphones
Heterodyne Detector
Wave Analyzer
Int.

FIG. (8)

spectrum shape so that amplitude changes by -6 dB per octave. This spectrum shaping may be obtained with single resistor-capacitor high-pass and low-pass circuits.

The four non-linear devices were constructed with both transformers and operational amplifiers to reduce deviations from the ideal characteristics. The distorting device output was bandpass limited to 200-5000 Hz and either directly applied to the appropriate detector or frequency shaped by integration first. The appropriate detector was used for each respective signal type: Harmonic analyzer, headphones, heterodyne detector, and narrow band wave analyzer.

N.D. has an estimated uncertainty of ±7% of the tabulated values; whereas, H.D. can be measured with an uncertainty of ±1%. Uncertainty with noise measurements is introduced when the magnitudes of the two input signals, one without and one with a notch, are adjusted to be equal, and again when amplitudes are recorded following distortion. This involves, essentially, four amplitude measurements with noise signals. By contract H.D. requires at most two measurements of steady sine wave signals.

Subjective Effects

For distortion to be a useful measure of system fidelity, it must correlate to a subjective measure. Many subjective measures are possible, however, word intelligibility was selected as the measure which H.D. and N.D. might be expected and desired to predict.

Table II extracted from work of Licklider, et.al.[4,5,6] shows the monosyllabic word intelligibility of the distortion test devices for normal ears.

TABLE A-I

Device	Transfer Function	Output Autocorrelation Function
HWR	$V_{OUT} = V_{IN} \; : \; V_{IN} > 0$ $V_{OUT} = 0 \; : \; V_{IN} < 0$	$\frac{1}{2\pi}\left[\psi_j^2(0) - \psi_j^2(\tau) + \psi_j(\tau)\cos^{-1}\left(\frac{-\psi_j(\tau)}{\psi_j(0)}\right)\right]$
ASL	$V_{OUT} = \alpha V_{IN}^2 \; : \; V_{IN} > 0$ $V_{OUT} = -\alpha V_{IN}^2 \; : \; V_{IN} < 0$	$\frac{\psi_j^2(0)}{\pi}\left[\left(1 + 2\frac{\psi_j^2(\tau)}{\psi_j^2(0)}\right)\left(\pi - 2\cos^{-1}\left(\frac{\psi_j(\tau)}{\psi_j(0)}\right)\right) + 6\frac{\psi_j(\tau)}{\psi_j(0)}\sqrt{\frac{\psi_j^2(0) - \psi_j^2(\tau)}{\psi_j^2(0)}}\right]$
IPC	$V_{OUT} = E \; : \; V_{IN} > 0$ $V_{OUT} = -E \; : \; V_{IN} < 0$	$\frac{2}{\pi}\left[E^2 \sin^{-1}\left(\frac{\psi_j(\tau)}{\psi_j(0)}\right)\right]$
FWR	$V_{OUT} = \lvert V_{IN} \rvert$	$\frac{2\psi_j(0)}{\pi}\left[\left(1 - \frac{\psi_j^2(\tau)}{\psi_j^2(0)}\right)^{1/2} + \frac{\psi_j(\tau)}{2\psi_j(0)}\left(\pi - \cos^{-1}\left(\frac{\psi_j(\tau)}{\psi_j(0)}\right)\right)\right]$
SL	$V_{OUT} = B V_{IN}^2$	$\psi_j^2(0) + 2\psi_j^2(\tau)$

.TABLE II

Distortion Device	Intelligibility	Input Diff.	Output Int.	Combined
FWR	20%***			
HWR	98%**	+	-	+
IPC	80%** 92%*	96%*	92%*	96%*
ASL	60%	+	-	+

***Ref (4)

**Ref (5)

*Ref (6)

Published data are lacking for the specific crossover distorter, ASL, used in this investigation. We estimated the word intelligibility for ASL to be about 60%. This seems consistent with data reported by Licklider for sharp crossover distortion, i.e., center clipping. Differentiation of speech before distortiom improved the intelligibility for all devices, except for FWR which always remained poor; integration after distortion generally degraded the intelligibility of speech. The combined effect of differentiation before and and integration after distortion improved the intelligibility and often gave the most natural sounding system. Where quantitative intelligibility data are lacking, + signs in columns 3, 4 and 5 indicate that word intelligibility was observed to improve relative to the flat system of column 2; - signs indicate a deterioration. These data provide the basis for determining the effectiveness of distortion measures, N.D. and H.D., as predictors of intelligibility.

Harmonic Distortion

Table III shows the results of 500 cps H.D. measurements on the test devices. Distortion measured by N.D. will be compared with these H.D. measurements.

TABLE III

Distortion Device	H.D.	Input Diff.	Output Int.	Combined
FWR	100%	100%	100%	100%
HWR	42%	42%	22%	22%
IPC	31%	31%	12%	12%
ASL	21%	21%	7%	7%

Measured results include the first three largest harmonics. Our calculations for the ideal devices are close to the measured values. H.D. values shown are independent of drive level since the devices produce amplitude independent distortion. According to intelligibility score, the order of preference, decreasing intelligibility score, was HWR, IPC, ASL, FWR, while the H.D. order, increasing per cent distortion, is ASL, IPC, HWR, FWR. Relative ranking of HWR and PC is reversed and ASL certainly did not give the best intelligibility as H.D. measurements would indicate. Input differentiation improved intelligibility but H.D. remained constant. Output integration reduced intelligibility while H.D. predicted improvement. The combined effect predicts the intelligibility improvement. Many experimenters report that H.D. is an inconsistent predictor of word intelligibility and results in Table III support this conclusion.

Noise Distortion

Table IV shows the distortion measured with a noise signal using the heterodyne and passive filter methods.

TABLE IV

Distortion Device	Noise Distortion @ 2 KC Heterodyne	 Filter	Calculated Value for Filter Method (See Appendix I)
FWR	100%	100%	100%
HWR	42%	53%	49%
IPC	32%	40%	37%
ASL	28%	40%	34%

Test spectra for both cases were flat. Calculated values in Table IV were derived by approximate methods outlined in Appendix I. Fawe[9] recently gave an analytical result for an IPC from which N.D. ≈ 40% can be derived. The values obtained by the heterodyne method are less than those obtained by the filter method except where both equal 100%. There is better agreement between calculated and measured N.D. using the passive filter than by using the heterodyne generated notch noise. The devices are rank ordered the same in flat systems as by H.D. Thus N.D. is also inadequate as a predictor of word intelligibility in flat systems.

Table V gives test results using the Burnett method.

TABLE V

Distortion Device	"Noise Distortion"	Input Diff.	Combined Input Diff. and Output Int.
FWR	100%	100%	100%
HWR	58%	48%	56%
IPC	42%	36%	38%
ASL	45%	34%	33%

The test noise spectrum was shaped to approximate long term average speech. Ratios of distortion output to total output signal in dB at ten frequencies were averaged and converted to %. The frequencies were 200, 400, 600, 800, 1000, 1250, 1550, 2000, 2500 and 3000 Hz. Referring to the intelligibility scores of the devices, FWR is ranked as the worst intelligibility and highest distortion; but HWR, the best in intelligibility, has the second highest distortion. Within the limits of accuracy ASL and IPC have the same N.D. but differ significantly in word intelligibility. N.D. predicts the increased intelligibility due to input differentiation but different device ordering relative to intelligibility scores remains.

Since output integration affects distortion products and total signal identically, N.D. showed no sensitivity to output filtering while subjective speech intelligibility did. The combined effect on N.D. of input differentiation and output integration is identical to differentiation only.

N.D. vs. Frequency

Observations for the four devices tested showed:

1. Flat input test spectra produce nearly flat distortion spectra and flat total response spectra.
2. Differentiation or integration of input test spectra (±6 dB/octave) produced nearly flat distortion spectra, while total response spectra followed the frequency shaping.
3. Post distortion frequency shaping affected noise distortion spectra and total response spectra the same.
4. If the output passband, i.e., after distortion, extends above and/or below input passband, i.e., before distortion, N.D. approaches 100% at frequencies above and below the input passband.

The effects of spectrum shaping on H.D. and I.M. are well documented in the literature on distortion measurement (Cf. Refs. 7, 8).

Conclusions:

The H.D., I.M. and N.D measurements all give the ability of a system to create new frequencies, i.e., distribute signal energy to other parts of the spectrum. Mathematically, all three give the same rank ordering of distortion in simple non-linear elements. Experimentally, both H.D. and N.D. give the same rank ordering to the non-linear distortion devices considered here. The distortion ordering is not the same as the word intelligibility ordering. Apparently, measurement of non-linearity as evidenced by generation of new frequencies is not adequate for prediction of word intelligibility.

When spectrum shaping is included in the distorting system, N.D. and H.D. give different results which can be explained by emphasis of frequencies in the test signals. Nevertheless, neither distortion measurement gives values consistent with the subjective ordering by word intelligibility scores. Conclusions derived are limited to correlation with word intelligibility. It is not known how other subjective tests will correlate with these distortion measures. The usefulness of present distortion measurements, H.D., I.M., and N.D., is limited to rough indications of system quality. Thus, at this time there seems to be no advantage for a distortion measurement more complicated than H.D which uses a single sine wave as the test signal and a wave analyzer or a distortion meter that can be varied to obtain distortion as a function of frequency.

Reduction of non-linear distortion in a communication system is usually desirable. Any of the distortion measures can be used as an indicator of non-linearity in a given system. The basic difficulty with use of distortion as a predictor of speech intelligibility shows up when systems having different forms of non-linearity are compared. Then, none of the methods are suitable to give relative intelligibility ranking among a mixed group of non-linear systems.

REFERENCES

[1] B. E. Bang, "A.F. Non-linear Distortion Measurement by Wide Band Noise," N12 Fourth International Congress on Acoustics, Copenhagen, Aug. 1962.

[2] S. B. Pedersen, "Some Results on Using a Signal with a Continuous Spectrum for Distortion Measurements," N45 Fourth International Congress on Acoustics, Aug. 1962.

[3] E. D. Burnett, "A New Method for the Measurement of Non-linear Distortion Using a Random Noise Test Signal," NBS Report 9421 (Sept. 1966).

[4] J. C. R. Licklider, "Effects of Amplitude Distortion Upon the Intelligibility of Speech," J. Acoust. Soc. Am. 18, 429-434 (1946).

[5] J. C. R. Licklider and R. Held, "Effects of Various Types of Non-linear Distortion Upon the Intelligibility of Speech," J. Acoust. Soc. Am. 24, 114 (1952).

[6] J. C. R. Licklider and I. Pollack, "Effects of Differentiation Integration and Infinite Peak Clipping Upon the Intelligibility of Speech," J. Acoust. Soc. Am. 20, 42-51 (Jan. 1948).

[7] F. E. Terman, J. M. Pettit, Electronic Measurements (McGraw-Hill Book Company, Inc., New York, 1952) 2nd ed., pp. 334-341.

[8] F. Langford-Smith, Editor, Radiotron Designers Handbook (Wireless Press, 1953) 4th ed., pp. 603-634.

[9] A. L. Fawe, "Interpretation of Infinitely Clipped Speech Properties," IEEE Trans. Audio and Electroacoustics, AU-14, No. 4, 178-183 (Dec. 1966).

[10] S. O. Rice, "Mathematical Analysis of Random Noise," Bell System Tech. J., 23, 282 (1944).

[11] D. Middleton, "Some General Results in the Theory of Noise Through Non-linear Devices," Quart. Appl. Math, 5, No. 4, 445-500 (Jan. 1948).

[12] Y. W. Lee, "Statistical Theory of Communication," (John Wiley & Sons, New York, 1960), Chapter 7.

[13] J. F. Barrett and D. G. Lampard, "An Expansion for Some Second Order Probability Distributions and Its Application to Noise Problems," Inst. Rad. Engrs. Trans. Information Theory 1, p. 10 (1955).

APPENDIX I - Calculating the Response of Non-linear Elements to Random Gaussian Noise.

This appendix contains an analysis of the noise distortion products generated by a Gaussian noise in a nonlinear system representable by a power series. It shows that (1) the N.D. is independent of spectrum shaping following a nonlinear device; (2) outside the pass band N.D. approaches 100%; (3) N.D. depends on the frequency of observation; and (4) N.D. is a measure of the curvature of the transfer function. These results apply to the passive filter method of N.D. measurement.

Fig. A1 illustrates a model suitable for calculating the experimental results for N.D.

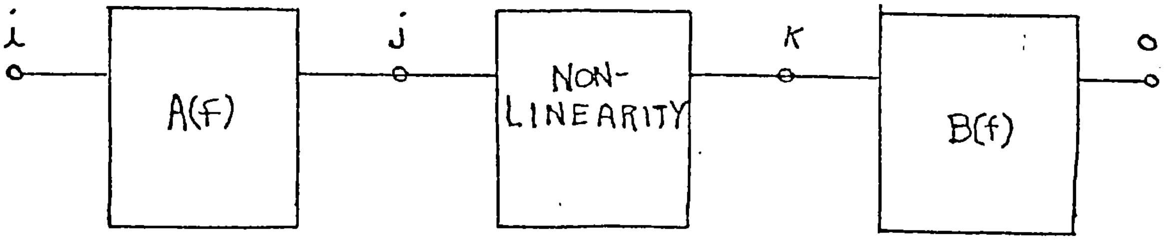


FIG. A1

The system consists of a linear but frequency dependent input circuit having transfer function A(f); followed by a memory free frequency independent nonlinear element; followed by a linear but frequency dependent output circuit with transfer function B(f). Assuming a Guassian noise input whose power spectrum is $W_i(f)$, the output power spectrum $W_o(f)$ is required.

Since (1) $$W_j(f) = W_i(f)\,|A(f)|^2$$

and (2) $$W_o(f) = W_k(f)\,|B(f)|^2$$

the problem reduces to finding power spectrum $W_k(f)$ at k in terms of power spectrum $W_j(f)$ at j. The effect of a nonlinear device on noise signals has been studied by several investigators[2,10,11] and two analytical methods have been developed.

The Direct Method[10]

A sum of sine waves each having random phase angle ϕ_m is used to represent the voltage at j.

(3)
$$V_j = \sum_{-\infty}^{+\infty} C_m(f)\, e^{i\omega_m t - i\phi_m}$$

The series for V_j is operated on according to the law of the nonlinear device. For instance, a square law device requires squaring the series. The contribution from each term in the series in each frequency increment Δf_m is summed to construct the output spectrum. Rice[10] has detailed this technique. Although the method has intuitive appeal, formulation is long and tedious for all but the simplest nonlinear functions.

Correlation Method[10]

When direct calculations of the output spectrum proves unwieldy, calculation of the output autocorrelation function gives a simplifying intermediate step. The output spectrum is then obtained from the autocorrelation function with the Wiener autocorrelation theorem.

The autocorrelation function is the time average of the product of the signal at time t and the signal at time $t+\tau$. for V_j :

(4)
$$\Psi_j(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_0^T V_j(t)\, V_j(t+\tau)\, dt$$

The autocorrelation function depends on the delay time:

When $\tau \to 0$: $\psi_j(\tau) \to \overline{V_j^2}$ the mean square voltage at j.

When $\tau \to \infty$: $\psi_j(\tau) \to 0$ for random processes.

The autocorrelation function is related to the power spectrum by cosine transforms:

$$W_j(f) = 4\int_0^\infty \psi_j(\tau) \cos 2\pi f \tau \, d\tau \tag{5}$$

$$\psi_j(\tau) = \int_0^\infty W_j(f) \cos 2\pi f \tau \, df \tag{6}$$

Example: Assume a power series represents the nonlinear element transfer function, where we will consider only the first two terms:

$$V_K = a V_j + b V_j^2 + c V_j^3 + \cdots \tag{7}$$

Let $W_i(f)$ be a wideband white noise spectrum and A(f), B(f) arbitrary frequency shaping.

$$W_j(f) = W_i(f) \, |A(f)|^2 \tag{8}$$

$$\psi_j(\tau) = \int_0^\infty |A(f)|^2 W_i(f) \cos 2\pi f \tau \, df \tag{9}$$

Eq. 9 does not require explicit evaluation of $\psi_j(\tau)$ but establishes the relationship between the spectrum and its correlation function. The autocorrelation function at the output of the nonlinear device is:

$$\psi_K(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T V_K(t) \, V_K(t+\tau) \, dt \tag{10}$$

Since the signal is a random function in time, Eq. 10 cannot be calculated from explicit time functions (i.e., function relationships in time do not exist for random functions). Thus Eq. 10 cannot be evaluated as written. Since $\psi_K(\tau)$ is averaged over all time and $V_k(t)$ is a random function the time average can be replaced by an ensemble average.[12]

For Gaussian statistics the function $P(V_1, V_2, \tau)$ gives the probability that V lies in a range of V_1 to $V_1 \pm dV_1$ at a time t while V also lies in a range V_2 to $V_2 \pm dV_2$ at a later time $t + \tau$. $\psi_k(\tau)$ can be evaluated using $P(V_1, V_2, \tau)$ as the weighting function.

$$\psi_k(\tau) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (aV_1 + bV_1^2)(aV_2 + bV_2^2)\, P(V_1, V_2, \tau)\, dV_1\, dV_2 \tag{11}$$

Eq. (11) is evaluated by using Mehler's expansion[13] for $P(V_1, V_2, \tau)$ and writing the power series as a sum of Hermite polynomials. Pederson used this method[2]. Other methods include a characteristic function method.[10,11] The best technique is usually indicated by the type of functional dependence of the transfer characteristic.

Evaluation of Eq. 11 yields:

$$\psi_K(\tau) = a^2 \psi_j(\tau) + b^2\left[\psi_j^2(0) + 2\psi_j^2(\tau)\right] \tag{12}$$

In general, the result for $\psi_K(\tau)$ can always be expanded in a power series of $\psi_j(\tau)$.

$$\psi_K(\tau) = \sum_n C_n \psi_j^n(\tau)$$

Since $W_k(f)$ is the cosine transform of $\psi_K(\tau)$, integrals of the type below must be evaluated.

$$(13)\qquad G_n(f) = \int_0^{\infty} \psi_j^n(\tau) \cos 2\pi f\tau \, d\tau \qquad n = 0, 1, 2 \cdots$$

Using Eq. (5)

$$(14)\qquad W_j(f) = 4 \int_0^{\infty} \psi_j(\tau) \cos 2\pi f\tau \, d\tau$$

and the established relationship[10]:

$$(15)\qquad G_n(f) = \frac{1}{2} \int_{-\infty}^{+\infty} \omega_j(f-x)\, G_{n-1}(x) \, dx$$

where $\omega_j(f) = \omega_j(-f) = W_j(f)$

$W_k(f)$ will be established through a set of convolution integrals of $W_j(f)$.

For this example (ignoring d.c. terms):

$$(16)\qquad W_k(f) = a^2 W_j(f) + b^2 \int_{-\infty}^{+\infty} \omega_j(f-x)\, \omega_j(x) \, dx + \cdots$$

The spectrum contains a linearly transferred term $a^2 W_j(f)$ plus higher order distortion products included in the integral. The N.D. is expressed as:

$$(17)\qquad \text{N.D.} = 100 \times \left[\frac{b^2 \int_{-\infty}^{+\infty} \omega_j(f-x)\, \omega_j(x)\, dx + \cdots}{a^2 W_j(f) + b^2 \int_{-\infty}^{+\infty} \omega_j(f-x)\, \omega_j(x)\, dx + \cdots} \right]^{1/2}$$

Note that the post distortion frequency term B(f) will not be present in the ratio because the transfer function B(f) operates on both the distortion and linear products identically. Thus, we have shown that N.D. is independent of post distortion filtering.

Consider further that if A(f) is flat and unity at all frequencies then

$$W_j(f) = W_i(f)$$

Let $W_i(f)$ be uniform from 0 to cut-off at f_μ and have a power density W_I in this band, then

$$(18) \quad \text{N.D.} = 100 \times \left[\frac{1}{1 + \dfrac{a^2 W_i(f)}{b^2 W_I^2 (2f_\mu - f)}} \right]^{1/2}$$

Outside the pass band $W_i(f) \rightarrow 0$ and therefore the N.D. → 100% since no linear components exist.

Within the pass band $W_i(f) = W_I$

$$(19) \quad \text{N.D.} = 100 \times \left[\frac{1}{1 + \dfrac{a^2}{2b^2 V_I^2 \left(1 - \frac{f}{2f_\mu}\right)}} \right]^{1/2} ; \quad V_I^2 = W_I f_\mu$$

The N.D. is very weakly dependent on frequency and is determined by the ratio of a/b. Thus N.D. is a measure of the curvature of the nonlinear function.

The theoretical results in Table III of the text are calculated using the output autocorrelation functions in Table A-I. These functions are determined by evaluating Eq. (11) for the respective transfer characteristics.

Each of these functions was expanded in a power series of $\psi_j(T)$ and only the first few terms were evaluated. The manipulative steps follow as in Eq. (12) through (17). Results for noise distortion for each device will be of a form similar to Eq. (17) but with different coefficients and inclusion of some higher order terms.